This dist. of functions of RVs, expectation of sums and products, variance, SD; other moments of the dist.

Silicon Valley companies give signing bonuses as incentives to accept their job offers. These are often in the form of stock options: an opportunity to buy $N$ shares of the company one year from now at a price $S_0$ in your view. If the stock is likely to rise over the next year, you'll be able to sell at a profit. Define $X = (S_1 - S_0)$ (price of the stock 1 year from now).

For simplicity, pretend $X$ is discrete.
with only 2 values: \( X_1 < S \) and \( X_2 > S \), let \( p = P(X = X_2) \), the prob. that the stock will rise in value. You'd like to evaluate these stock options (e.g., to compare one company's job offer with that of another), but (of course) you don't know \( X \). Let \( I \) = value of option for one share at \( \$1 \) per share from now. If \( (X = X_1 < S) \), the option is worthless, and \( I = 0 \); otherwise (ignoring dividends & costs of buying & selling stocks) if \( (X = X_2 > S) \) then the option is worth \( (X_2 - S) \); thus \( I = h(X) = \begin{cases} 0 & \text{if } X = X_1 \\ (X_2 - S) & \text{if } X = X_2 \end{cases} \). To see how valuable the option is, you have to compare...
it to the return you would have received if you had not exercised the stock option; a reasonable point of comparison would be to invest in a bond that pays \( 2\% \) per year.

A fair measure of worth of the option would be the present value of \( V \), defined to be the number \( c \) such that

\[
E(\sqrt{V}) = (1 + d) \cdot c.
\]

But we already know

\[
E(\sqrt{V}) = \theta \cdot (1-p) + (x_2 - \delta^s) \cdot p = (x_2 - \delta^s) \cdot p,
\]

so \((1 + d) \cdot c = (x_2 - \delta^s) \cdot p\) and \( c = \left( \frac{x_2 - \delta^s}{1 + d} \right) \cdot p \).

to finish the calculation you need to specify \( p \). The standard way to do this
in the financial sector is to assume that the present value of \( X \) is equal in expectation to the current value of the stock price; i.e., to assume that the expected value of \((\text{buying 1 share & holding it for a year}) = (\text{investing the same amount of money in the risk-free alternative})\) — i.e., \( E(X) = (1 + \alpha) \cdot S' \).

But we already know that
\[
E(X) = p \cdot x_2 + (1-p) \cdot x_1 = (1 + \alpha) S';
\]
solving for \( p \) gives
\[
p = \frac{x_1 - (1 + \alpha) S'}{x_1 - x_2} = \frac{(1 + \alpha) S' - x_1}{x_2 - x_1}.
\]

So the fair price \( C \) of an option to buy
The share is given by:

\[
c = \left( \frac{x_2 - S'}{1 + d} \right) \left[ \frac{(1+d)(S' - x_1)}{x_2 - x_1} \right].
\]

If we as an illustration:

- Downside: $20 (-10\%)
- Upright: $60 (430\%)

\(d = 0.04\) a realistic in 2001 or so but not today: \(d = 0.1\) now or 2.

With these values, \(c = 20.09\) (about 10\% of the current value of the stock). \(c\) is called the risk-neutral price of the option; under the assumptions made here, you could now sell the option today (if you had it) at a fair price of about $20; this would make you an options trader.

An investment that allows people to buy or sell an option on a security is called a derivative. (e.g. stock)
This is covariance, variance, correlation.

You have $1,000 to invest.

Suppose for simplicity that you want to build a portfolio consisting of (some shares of company A, some shares of company B, some fixed-rate bonds). You need to figure out the optimal allocation across these 3 assets: how much A, how much B, how much C. Best portfolio depends...
on how much the stock prices of A and C will change over a reasonable time period (say, 1 year); these changes are unknown, so you model them probabilistically.

For company A stock,

\[ R_A = \text{rate of return (change in price) of a single share} \]

\[ R_C = \text{fixed} \]

Bonds B will appreciate at a fixed rate R.

Your portfolio is (5A shares of A stock, 5C shares of C stock, $50 worth of bonds)

1-year return on your portfolio is

\[ \left( \frac{5A R_A + 5C R_C + \$50}{5} \right) \]
Define $E(R_A) = \mu_A$, $V(R_A) = \sigma^2_A$

$E(R_c) = \mu_c$, $V(R_c) = \sigma^2_c$

Constraint on $(S_A, S_c, S_B)$: total current value of portfolio = $I$

$P_A$ = current price of stock A

$P_c$ = current price of stock C

Then the constraint is

$S_A P_A + S_c P_c + S_B = I$

(plus one of course, $S_A \geq 0$

like $S_c \geq 0$

$S_B \geq 0$)

For now, pretend simplistically that $P_A$ and $P_c$ are independent;

then the mean and variance of the return on your portfolio are as follows:
\[ E(R_A + R_c + \epsilon) = \mu_A + \mu_c + \epsilon \]
\[ V(R_A + R_c + \epsilon) = \sigma_A^2 + \sigma_c^2 \]

Obviously, you want the expected return to be high; to minimize your downside risk, you want the variance to be low. Simultaneously maximizing \( \max E \) and minimizing \( \min V \) is rarely possible.

You need, in practice, to sacrifice some expected return to keep your downside risk small. How should you make this trade-off?
subject to: \[
E = \sum a_i E_i + 0.025 = 1
\]
\[
\sum a_i + \sum b_i = 0
\]
\[
\sum a_i^2 + \sum b_i^2 = 0.25
\]
\[
s_{a_i} \geq 0, s_{b_i} \geq 0
\]

For fixed return - \( \mu \), you want one of these.

For fixed return, we choose \( \sigma^2 \) and \( \mu \), then return are called efficient.

With minimum variance for a fixed mean return, the smallest variance is best - portfolio lies on the line of the mean return with \( \sigma = 0 \). Any portfolio above lies on the mean-variance set of lines. The one with the highest return, for example, is your expected return.

Surprise you set return.
This min problem can be solved with a calculus technique called Lagrange multipliers (attributed to the Italian mathematician who went by the French name Joseph-Louis Lagrange (1736-1813)).

See IS pp. 230-231 for a numerical example.

Note: The optimal portfolio in their example with $I = \$100,000$ and $T = \$7,000$ (7% return) of return (reasonable) has variance $2.55 \cdot 10^{-2} \text{ spare}$, which means an SD of return of $\sqrt{2.55 \cdot 10^{-2}} = \$5050$.

We'll see later that if the dist. of return follows the bell curve, you would have about an 8% chance of losing money.