~~This~~ ~~last~~ ~~time: functions of rvs, expectation~~

read: DS ch. 4 pp. 215 - 260

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Case study: option pricing

~~next~~ This time: expectation of sums and products, variance, SD; other moments of the dist.

Silicon valley companies give signing bonuses as incentives to accept their job offers. These are often in the form of stock options: an opportunity to buy N shares of the company one year from now at a (known) price, S. If in your view the stock is likely to rise over the next year, you'll be able to sell at a profit. Define X = (price of the stock 1 year from now). For simplicity pretend X is discrete with only 2 values: $x_1 < \mathcal{S}$ and $x_2 > \mathcal{S}$, let $p = \underline{P}(\underline{X} = x_2)$, the prob. that the stock will rise in value. You'd like to evaluate these stock options (e.g., to compare one company's job offer with that of another), but (of course) you don't know $\underline{X}$. Let $\underline{Y}$ = value of option for one share at \$$\mathcal{S}$ 1 year from now.

If ($\underline{X} = x_1 < \mathcal{S}$), the option is worthless and $\underline{Y} = 0$; otherwise (ignoring dividends & costs of buying & selling stocks) if ($\underline{X} = x_2 > \mathcal{S}$) then the option is worth $(x_2 - \mathcal{S})$; thus

$$\underline{Y} = h(\underline{X}) = \begin{cases} 0 & \text{if } \underline{X} = x_1 \\ (x_2 - \mathcal{S}) & x_2 \end{cases}.$$

To see how valuable the option is, you have to compare it to the return you would have received if you had not "exercised" the stock option; a reasonable point of comparison would be to invest in a bond or other fixed security that pays α%/year.

A fair measure of worth of the option would be the present value of Y, defined to be the number c such that

$E(Y) = (1+\alpha)\cdot c$. But we already know

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that $E(Y) = 0\cdot(1-p) + (x_2 - S)\cdot p = (x_2 - S)p$,

so $(1+\alpha)\cdot c = (x_2 - S)\cdot p$ and $c = \left(\frac{x_2 - S}{1+\alpha}\right)\cdot p$.

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to finish the calculation you need to specify p. The standard way to do this

in the financial sector is to assume that (4)
the present value of X is equal in
expectation to the current value of the
stock price: i.e., to assume that the
expected value of (buying 1 share &
holding it for a year) = (investing the
same amount of money in the risk-free
alternative) — i.e., $E(X) = (1+\alpha) \cdot S$.

But we already know that

$E(X) = p \cdot x_2 + (1-p) \cdot x_1 \stackrel{\Delta}{=} (1+\alpha) S$; solving
for p gives $p = \dfrac{x_1 - (1+\alpha) S}{x_1 - x_2} = \dfrac{(1+\alpha) S - x_1}{x_2 - x_1}$.

So the fair price (c) of an option to buy

one share is given by
$$c = \left(\frac{x_2 - S}{1 + \alpha} \right) \left[\frac{(1+\alpha) S - x_1}{x_2 - x_1} \right].$$

DS use as illustration
$S = \$200$
$x_1 = \$180$
$x_2 = \$260$
$\alpha = .04$

downside $20 (-10%)

upside $60 (+30%)

realistic in 2001 or so but not today: $\alpha \doteq$ .01 or .02 now

With these values $c \doteq \$20.19$ (about 10% of the current value of the stock).

c is called the risk-neutral price of the option; under the assumptions made here, you could now sell the option today (if you had it) at a fair price of about $20; this would make you an options trader.

An investment that allows people to buy or sell an option on a security (e.g. stock) is called a derivative.

~~This time: expectation, variance, standard deviation, moments~~

This ~~week~~ time: covariance, correlation, conditional expectation, utility

read: DS pp. 248-274

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Case study: Building a good portfolio

You have \$ I to invest.

Suppose for simplicity that you want to build a portfolio consisting of (some shares of company A, some shares of company C, some fixed-rate bonds B). You need to figure out the optimal allocation across these 3 assets: how much A, how much C, how much B? Best portfolio depends on how much the stock prices of A and C will change over a reasonable time period (say, 1 year); these changes are unknown, so you model (your uncertainty about) them probabilistically.

$R_A \triangleq$ rate of return (change in price) of a single share	for company A stock
$R_C \triangleq$ ______	C

Bonds B will appreciate at a fixed yearly rate r.

your Portfolio = (s_A shares of A stock, s_C shares of C stock, \$ s_B worth of bonds)

1-year return on your portfolio is

fixed — random — fixed

$$\left(\underbrace{s_A R_A}_{\$} + \underbrace{s_C R_C}_{\$} + \$ \, r \, s_B \right)$$

Define $E(R_A) = \mu_A$, $V(R_A) = \sigma_A^2$ (3)

$$E(R_c) = \mu_c, \quad V(R_c) = \sigma_c^2$$

Constraint on (s_A, s_c, s_B): total current value of portfolio $= \$I$;

P_A = current price of stock A

P_c = ——————— C

then the constraint is

$$s_A P_A + s_c P_c + s_B = I$$

(plus, of course, integers $s_A \geq 0$, $s_c \geq 0$, $s_B \geq 0$)

For now, pretend simplistically that R_A and R_c are independent; then the mean and variance of the return on your portfolio are as follows.

$$E(s_A R_A + s_C R_C + r s_B) = s_A \mu_A + s_C \mu_C + r s_B$$

random

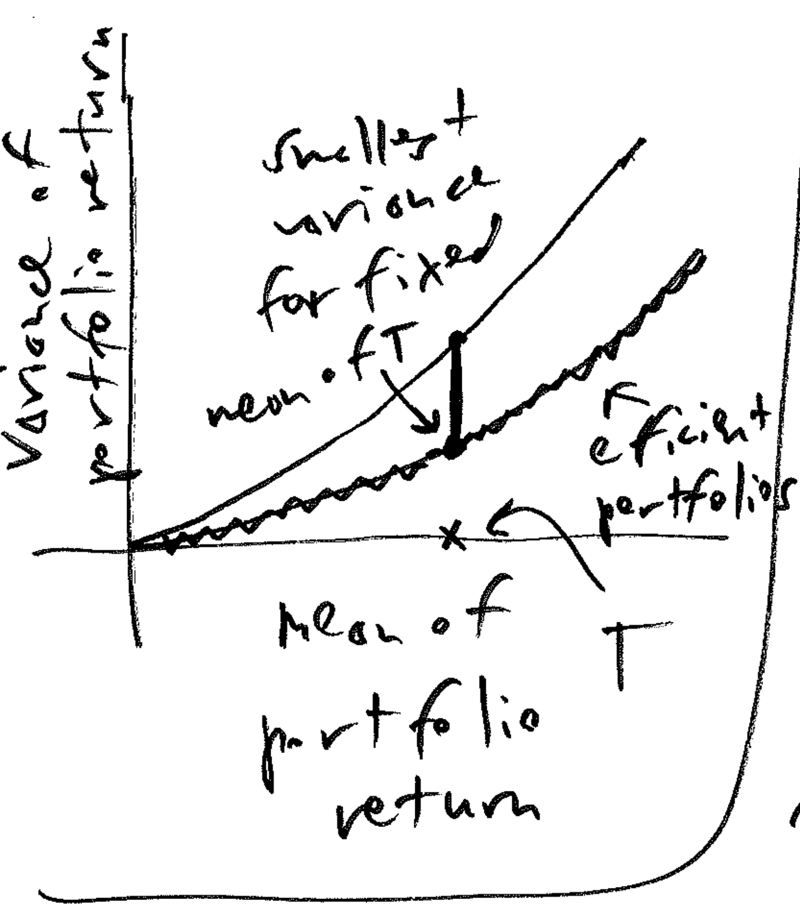
$$V(s_A R_A + s_C R_C + r s_B) \overset{\text{indep}}{=} s_A^2 \sigma_A^2 + s_C^2 \sigma_C^2$$

constant

obviously you want the expected return to be high; to minimize your downside risk, you want the variance to be low.

Simultaneous $\begin{pmatrix} \max E \\ \min V \end{pmatrix}$ rarely possible.

You need, in practice, to sacrifice some expected return to keep your downside risk small. How should you make this trade-off? optimally



Suppose you set a target \$T for your expected return. Among all portfolios with E(return) = T, the one with the smallest variance is best — portfolios with minimum variance for a fixed mean return are called <u>efficient</u>, & you want one of these.

Math problem:

For fixed known $\left(\underset{?}{\overset{P_A}{\mu_A}}, \underset{?}{\overset{P_c}{\mu_c}}, \underset{?}{\sigma_A^2}, \underset{?}{\sigma_c^2}, \overset{T,\ I}{r}\right)$, find (s_A, s_c, s_B) to minimize $V = s_A^2\sigma_A^2 + s_B^2\sigma_c^2$

subject to
$$\left\{\begin{array}{l} E = s_A\mu_A + s_c\mu_c + r s_B = T \\ \text{and} \\ s_A P_A + s_c P_c + s_B = I \end{array}\right. \quad \left(\begin{array}{l} s_A \geq 0 \\ s_B \geq 0 \\ s_c \geq 0 \end{array}\right)$$

This math problem can be solved with a calculus technique called <u>Lagrange multipliers</u> (attributed to the Italian mathematician who went by the French name Joseph-Louis Lagrange (1736-1813)).

See DS pp. 230-231 for a numerical example.

<u>Note:</u> The optimal portfolio in their example with $I = \$100{,}000$ and $T = \$7{,}000$ (7% return; reasonable) has variance of return $2.55 \cdot 10^7 \, \$^2$, which means an SD of return of $\sqrt{2.55 \cdot 10^7} \doteq \$5050$; we'll see later that if the dist. of return follows the bell curve, you would have about an 8% chance of losing money.