AMS 131
31 Jul 19

this time: **and**, **or**, **not**
set theory

next time: foundations

office hours starting today

$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$

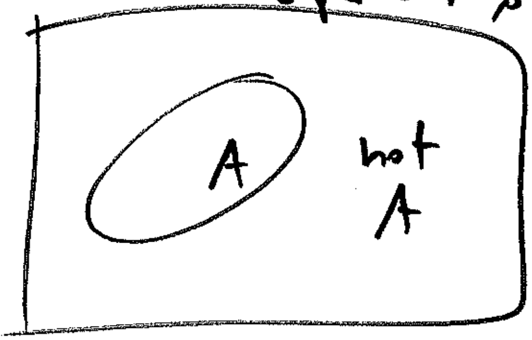
if $P(A \text{ and } B) = 0$, A, B said to be **mutually exclusive**

$0 \le P(A) \le 1$

"impossibility" — A False

"certainty" — A True

sample space S



$P(S) = 1$

$P(A \text{ or } \text{not} A) = 1$

$1 = P(A) + P(\text{not} A)$

$P(A) = 1 - P(\text{not} A)$

direct (difficult)

indirect (easy)

and

pop.

$$\begin{bmatrix} 1 \\ 2 \\ 7 \end{bmatrix} \xrightarrow{\text{at random}} \begin{matrix} Y_1 \\ Y_2 \end{matrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

random variable → Y_1, Y_2; sample; value → y_1, y_2

$n = 2$

$$P_{IID}(Y_1 = 7 \text{ and } Y_2 = 7) = ?$$

Case I: at random with replacement (IID: independent identically distributed)

$S = \{(1,1), \ldots, (7,7)\}$

2nd draw; 1st draw; (E o s)

1st draw \ 2nd draw	1	2	7
1	(1, 1)	(1, 2)	(1, 7)
2	(2, 1)	(2, 2)	(2, 7)
7	(7, 1)	(7, 2)	(7, 7)

3×3 contingency table

ELM? yes, by IID

$$P_{IID}(Y_1 = 7 \text{ and } Y_2 = 7) = \frac{1}{9}$$

$$P_{IID}(Y_1 = 7) = \frac{1}{3} = \frac{3}{9}$$

$$P_{IID}(Y_2 = 7) = \frac{1}{3} = \frac{3}{9}$$

conjecture

↓

$$P_{IID}(X_1 = 7 \text{ and } X_2 = 7) = \frac{1}{9}$$

$$= P_{IID}(X_1 = 7) \cdot P_{IID}(X_2 = 7)$$

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

case 2: at random without replacement (SRS: simple random sampling)

common sense $P_{SRS}(X_1 = 7 \text{ and } X_2 = 7) = 0$

2nd draw

1st draw	1	2	7
1	~~(1,1)~~	(1,2)	(1,7)
2	(2,1)	~~(2,2)~~	(2,7)
7	(7,1)	(7,2)	~~(7,7)~~

EOs

(ELM? yes ↳ SRS)

$$P_{SRS}(X_1 = 7 \text{ and } X_2 = 7) = \frac{0}{6} = 0 \checkmark$$

$$P_{SRS}(X_1 = 7) = \frac{1}{3} = \frac{2}{6}$$

$$P_{SRS}(X_2 = 7) = \frac{2}{6} = \frac{1}{3}$$

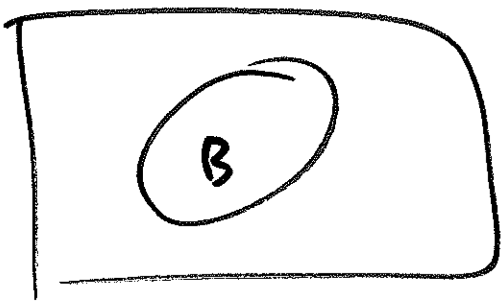
but

$$\underbrace{P_{SRS}(X_1 = 7 \text{ and } X_2 = 7)}_{= 0} \neq \underbrace{P_{SRS}(X_1 = 7)}_{1/3} \cdot \underbrace{P_{SRS}(X_2 = 7)}_{1/3}$$

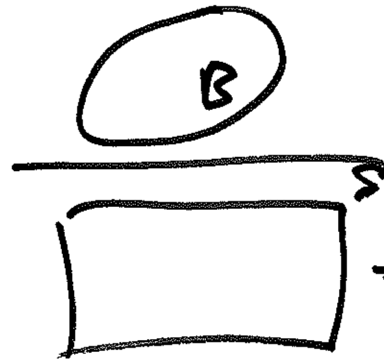
def. (1710) (A. de Moivre & T. Bayes (1750)

$$P(B \text{ given } A) = P(B \mid A)$$

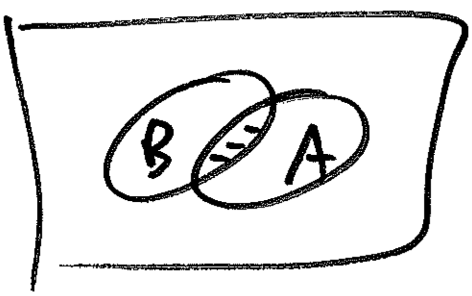
← "given"



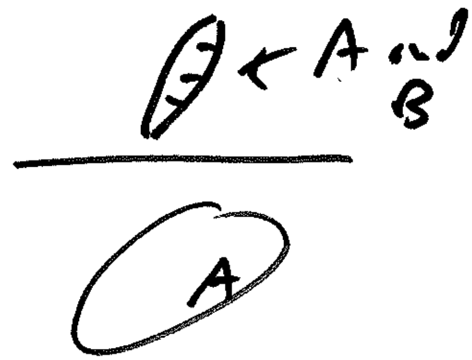
$P(B) =$



$= 1$



$P(B \mid A) =$



$$P(B \mid A) = \begin{cases} \dfrac{P(B \text{ and } A)}{P(A)} = \dfrac{P(A \text{ and } B)}{P(A)} & \text{if } P(A) > 0 \\ \text{undefined} & \text{if } P(A) = 0 \end{cases} \quad (9.56)$$

$\boxed{\text{if } P(A) > 0}$ $\quad P(B \mid A) = \dfrac{P(B \text{ and } A)}{P(A)} = \dfrac{P(A \text{ and } B)}{P(A)}$ (5)

$$\boxed{\begin{aligned} P(B \text{ and } A) &= P(A) \cdot P(B \mid A) \\ \shortparallel \quad & \qquad \shortparallel \\ P(A \text{ and } B) &= P(B) \cdot P(A \mid B) \end{aligned}}$$

$$P(A \mid B) = \frac{P(A \text{ and } B)}{P(B)} \quad \left(\text{if } P(B) > 0\right)$$

general product rule for and

$$P_{SRS}(Z_1 = 7 \text{ and } Z_2 = 7)$$

$$= P_{SRS}(Z_1 = 7) \cdot P_{SRS}(Z_2 = 7 \mid Z_1 = 7)$$

$$= \frac{1}{3} \cdot 0 = 0 \checkmark$$

Q: does general rule work for IID? 6

A: $P_{IID}(\mathcal{I}_1 = 7 \text{ and } \mathcal{I}_2 = 7)$

$$= P_{IID}(\mathcal{I}_1 = 7) \cdot P_{IID}(\mathcal{I}_2 = 7 \mid \mathcal{I}_1 = 7)$$

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but under at random with repl.

$$P_{IID}(\mathcal{I}_2 = 7 \mid \underline{\underline{\mathcal{I}_1 = 7}})$$

$$= P_{IID}(\mathcal{I}_2 = 7) = \frac{1}{3}$$

Bayes def. A, B are **independent** iff info. about B doesn't change prob. for A, & vice versa

Bayesian $\leftrightarrow$ information

[freq. def.] A, B independent iff

$$P(B \mid A) = P(B)$$

and $P(A \mid B) = P(A)$;

moreover ~~iff~~ A, B indep. iff

$$P(A \text{ and } B) = P(A) \cdot P(B \mid A)$$
$$= P(A) \cdot P(B)$$

$$P(B \text{ and } A) = P(B) \cdot P(A \mid B)$$
$$= P(B) \cdot P(A)$$

i.e., special case of product rule under independence:

A, B indep. iff $P(A \text{ and } B) = P(A) \cdot P(B)$

T-S case study

p(1 or more T-S babies in family of 5, both parents carriers)

$= 1 - p(\text{not } \underline{\qquad})$

$= 1 - p(0 \text{ T-S babies})$

$= 1 - p\left(\begin{matrix}\text{not}\\ \text{T-S}\\ \text{on}\\ 1^{st}\end{matrix} \text{ and } \begin{matrix}\text{not}\\ \text{T-S}\\ \text{on}\\ 2^{nd}\end{matrix} \text{ and } \cdot \text{ and } \begin{matrix}\text{not}\\ \text{T-S}\\ \text{on}\\ 5^{th}\end{matrix}\right)$

$\overset{\text{I ID}}{=} 1 - p\left(\begin{matrix}\text{not}\\ \text{T-S}\\ \text{on}\\ 1^{st}\end{matrix}\right) \cdot p\left(\begin{matrix}\text{not}\\ \text{T-S}\\ \text{on}\\ 2^{nd}\end{matrix}\right) \cdots p\left(\begin{matrix}\text{not}\\ \text{T-S}\\ \text{on}\\ 5^{th}\end{matrix}\right)$

$\overset{\text{IED}}{=} 1 - \left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{4}\right) \cdots \left(1 - \frac{1}{4}\right)$

$= 1 - \left(1 - \frac{1}{4}\right)^5 \doteq 76\%$

p 1 or more in n children

$\boxed{\text{PLAN AHEAD}}$

P(1 or more bad thing happening in n indep occurrences, P(bad thing on any single occurrence) = p

$$= 1 - (1-p)^n$$

Special case of Binomial probability

(ratio)

odds o in favor of A = $o = \frac{p}{1-p}$, $p = P(A)$

odds of 1 or more T-s bodies $= \frac{0.76}{1-0.76} = \frac{0.76}{0.24}$

$\doteq 3$

solve $o = \frac{p}{1-p}$ for p:

$$o(1-p) = p = o - op$$

$$p + op = o = p(1+o)$$

$$p = \frac{o}{1+o} \quad (10.57)$$

$S = \{0, 1\}$

$\emptyset$ null set

$\{0\}$

$\{1\}$

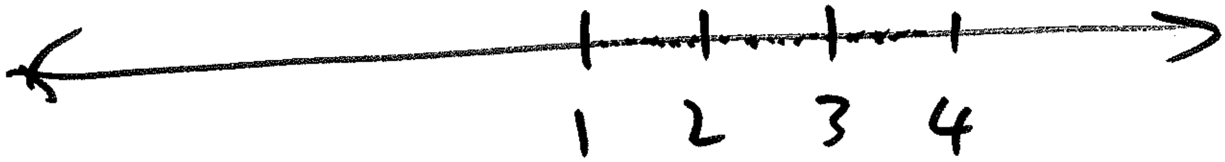
$\{0, 1\} = S$

$|S| = 2$

$|2^S| = 4 = 2^2$

$\{0, 1, \ldots, n\}$

$\{0, 1, 2, \ldots\}$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