08/02/19

Lecture 3 AMS 131

P(of being infected in 100 acts) = P(1 or more infections)

HIV → same as T-S case = 1 - P(no infections) = $P^*$

$p = (\frac{1}{500})$ $p = (\frac{1}{4})$ = 1 - P(not infected on first

n = 100 n = 5 AND not infected on second

... not infected on $100^{TH}$)

P(A and B) = P(A) · P(B) ← independent

independent part of IID:

$P^* = 1 -$ P(not infected on first) ·

P(not infected on second) · ...

P(not infected on $100^{TH}$)

$= 1 - (1 - \frac{1}{500})(1 - \frac{1}{500}) \dots (1 - \frac{1}{500})$

$= 1 - (1 - \frac{1}{500})^{100} = 0.18$

P(infected in 500 acts) $= 1 - (1 - \frac{1}{500})^{500}$

$= 0.63$

A, B independent iff

P(B | A) = P(B)

P(A | B) = P(A)

P(A and B) = P(A) · P(B)

A, B mutually exclusive

P(A and B) = 0

no overlap

$A \cap B = \emptyset$

can't both be true

P(inf. in partner $i$ in 1 act) = $p_i$

P(inf. in $n$ acts) $= 1 - (1 - p_1)(1 - p_2) \dots (1 - p_n)$

$= 1 - \prod_{i=1}^{n} (1 - p_i)$ $\frac{1}{1000} \leq p_i \leq \frac{1}{100}$

(product distribution)

pretend all $p_i = p$ right answer: $1 - (1-p)^n$

Dr. S: $np$

Wolfram alpha: $np - \frac{1}{2}((n-1)n)p^2 \dots$ Taylor expansion

value is always smaller than $np$

AMS 131 | Lecture 3 (cont.)

Extra notes (07/31/19)

orders of infinity   rational = integers

countably infinite if 1-to-1 to integers

irrational is one order of infinity than rational

ex: $\mathbb{N} = \{1, 2, 3 \dots\}$ : countable.

$\mathbb{R} = \{\text{all real numbers}\}$ = uncountable

measure theory: trying to make basic stuff rigorous like length, area, volume

but infinity is weird to become rigorous.


all points are uncountable.

Axiom of choice: If I have n uncountably infinite number of sets, I can make a new set by taking one element from each set.

Given a sphere, you can break up the sphere into a finite number of non-overlapping subsets, rotate them around, and assemble them into two identical copies of the original sphere.

↳ when S is infinite, the power set $2^S$ is "too big and strange" to permit the assignment of probabilities to all the $2^S$ sets in a logical, consistent way.

smaller set C to restrict to subsets of S

1. C includes entire sample space S
2. If event A is in C, so is its complement $A^c$
3. If $A_1, A_2 \dots A_n$ are all in C, then so is: $\bigcup_{i=1}^{\infty} A_i$   union: add up, no duplicates.

Lecture 3 (cont.) AMS 131

Whenever $|S| < \infty$ (finite) we can take $C = 2^S$,
so we can meaningfully assign probabilities:

$(A^c)^c = A$

$\emptyset^c = S$ and $S^c = \emptyset$

$A \cup A = A$ $\quad$ $A \cup B = B \cup A$

$A \cup A^c = S$ $\quad$ if $A \subset B$, then $A \cup B = B$

$A \cup \emptyset = A$

$A \cup S = S$ $\quad$ union ~ "or"

associative property: $A \cup B \cup C = A \cup (B \cup C) = (A \cup B) \cup C$

intersection ~ "and" $\quad$ $A^c \to$ not A
$A \cup B \to$ A or B
$A \cap B \to$ A and B $\quad$ only elements belonging to both sets

T-S case
$A = \{NNNNN\}$ as a set is equivalent to the true/false proposition: (exactly 0 T-S babies) being true

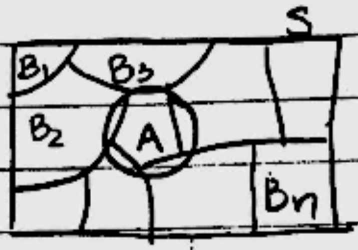
The intersection of multiple sets: $= \bigcap_{i=1}^{n} A_i$ has the same associative property

Two sets A, B, are mutually exclusive $\hat{=}$ disjoint
if $A \cap B = \emptyset$ (if they have no outcomes in common)
n sets are disjoint if all distinct pairs
are disjoint: $A_i \cap A_j = \emptyset$ for $i \neq j$
mutually exclusive $\leftrightarrow$ cannot both be true

De Morgan's Law $\quad$ $(A \cup B)^c = A^c \cap B^c$
$(A \cap B)^c = A^c \cup B^c$

distributive property $\quad$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

AMS 131 | Lecture 3 (cont.)




broke up sample space
- all mutually exclusive
- exhaustive, a point must be in one of the sections

$\{B_1, B_2, \dots B_n\}$ = partition of S.
every outcome in S lives in one (and only one) section of S.

The partitions also apply to A, where those sections are mutually exclusive and exhaustive

$A = (A \cap B_1) \cup (A \cap B_2) \dots \cup (A \cap B_n)$
$= (A \text{ and } B_1) \text{ or } (A \text{ and } B_2) \dots \text{ or } (A \text{ and } B_n)$

$P(A) = P[(A \text{ and } B_1) \text{ or } (A \text{ and } B_2) \dots \text{ or } (A \text{ and } B_n)]$
$= P(A \text{ and } B_1) + P(A \text{ and } B_2) + \dots + P(A \text{ and } B_n)$
$= \sum_{i=1}^{n} (A \text{ and } B_i)$ can sum them up, and there is no overlap!
$P(A \text{ and } B_i) = P(B_i) \cdot P(A \mid B_i)$
$= \sum_{i=1}^{n} P(B_i) \cdot P(A \mid B_i)$ Law of Total Probability (LTP) ★
- problem 4 on take home test #1

Kolmogorov Probability Axioms $P_K(A)$
$P_K(A)$ needs to be a function from C (the collection of non-weird subsets) to the real number line $\mathbb{R}$ - what else should be assumed about $P_K$?
$P_K(A): C \xrightarrow[\text{"maps to"}]{} [0,1]$ number (probability) between 0 and 1

Lecture 3 (cont.) AMS 131

Axiom 1: For all events $A \in C$, $P_K(A) \geq 0$
(motivated by relative frequency)

Axiom 2: $P_K(S) = 1$

Axiom 3: For every countable collection of disjoint events $A_1, A_2 \ldots \in C$

$$P_K\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P_K(A_i)$$ add up all of the relative probabilities

Axiom 3': equivalent to Axiom 3
if A is "close" to B, then $P(A) \doteq P(B)$ approximately

Consequences from Axioms
1. $P(\emptyset) = 0$
2. $P(A^C) = 1 - P(A)$

$P(A) = Pr(A)$ (book)

3. If $A \subset B$, then $P(A) \leq P(B)$
4. For all events $A$, $0 \leq P(A) \leq 1$
5. For all events $A, B$ $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
general addition rule "or" "and"
6. For any events $A_1, A_2, \ldots A_n$ $P\left(\bigcap_{i=1}^{\infty} A_i\right) \leq \sum_{i=1}^{n} P(A_i)$

$$P\left(\bigcap_{i=1}^{\infty} A_i\right) \geq 1 - \sum_{i=1}^{n} P(A_i^c)$$ useful in stats.

T-S case $P(Y=0) = P(\text{exactly } 0 \text{ T-S babies})$

$= P(\text{1st not T-S and 2nd not T-S}) \ldots$

$= \left[1 - P(\text{1st T-S baby})\right] \ldots \left[1 - P(\text{5th T-S baby})\right] = 1p^0(1-p)^5$

$P(TTTTT) = p^5 = 1p^5(1-p)^0$ $= (1-p)^5$ $p = \frac{1}{4}$

$P(Y=1) = 5p^1(1-p)^4$ $P(Y=2) = 10p^2(1-p)^3$

← 5 options for having 1 T-S baby PASCALS $\Delta$

Scanned by CamScanner

AMS 131 | Lecture 3 (cont.)

| # of T-S babies | $P(Y=y)$ | with $p=\frac{1}{4}$ |
|---|---|---|
| 0 | $1p^0(1-p)^5$ | 0.2373 |
| 1 | $5p^1(1-p)^4$ | 0.3955 |
| 2 | $10p^2(1-p)^3$ | 0.2637 |
| 3 | $10p^3(1-p)^2$ | 0.0879 |
| 4 | $5p^4(1-p)^1$ | 0.0146 |
| 5 | $1p^5(1-p)^0$ | 0.0010 |
| | 1 | 1.0000 |

$P(Y=y) = \square\, p^y(1-p)^{5-y}$

↑ binomial coefficients

pascal's triangle

1 5 10 10 5 1

Permutations and Combinations  $n = 52$ cards (no replacement)

8 ♡

52 51 50 49 48

$(52)\cdot(51)$ ways for the first two

$n(n-1)\ldots(n-k+1)$

$= 311{,}875{,}200$ permutations

The number of permutations of $n$ distinct things taken $k$ at a time is $P_{n,k} = n(n-1)\ldots(n-k+1)$

How many orderings of 52 cards are there?

52 51 50 49 ... 1

$(52)\cdot(51)\cdot(50)\cdot(49) \cdot\ldots\cdot (1) = 52! \doteq 8.1\times10^{67}$

$$P_{n,k} = \frac{n(n-1)\ldots(n-k+1)(n-k)!}{(n-k)!} = \frac{n!}{(n-k)!}$$

$0! \triangleq 1$

T-S case where $K=1$, one T-S baby

order does not matter! $\{T_1, N_1, N_2, N_3, N_4\}$

$n! = 5! = 120$ ways to order them

Now, $\{T, N, N, N, N\}$

1! to arrange T, 4! to arrange Ns

$$= \frac{5!}{(1!)(4!)} = 5 = \frac{n!}{K!(n-K)!}$$

Given a set with $n$ distinct elements, each distinct subset of size $K$ is a combination

$$C_{n,K} = \frac{n!}{K!(n-K)!} = \binom{n}{K}$$

N choose K

binomial coefficient

T-S case $P(Y=y) = \binom{n}{y} p^y (1-p)^{n-y}$ valid for all $y$, $n \geq 1$

P(at least 2 people share the same birth date)

- same month, day - not year

even birth rate from Jan 1 → Dec 31, no Feb 29

365 days a year

$K$ = # of people registered $K = 93$ (AMS 7)

$n^K$ equally likely outcomes $n = 365$

if no one has same birthday:

person 1 $n = 365$

2 $n = 364$

...

$K$ $n = n-K+1$

$$P_{n,K} = \frac{n!}{(n-K)!}$$

$P(A) = 1 - P(\text{not } A)$

$$P(A) = 1 - \frac{n!}{(n-K)!\,K!} = 1 - \frac{365!}{272!\,365^{93}} = 1 - \frac{(365)(364)\ldots(273)}{(365)(365)\ldots(365)}$$

$$= 0.999997$$