Lecture 3

$P(\text{of being infected in 100 acts}) = P(1 \text{ or more infections})$

HIV $\rightarrow$ same as T-S case $= 1 - P(\text{no infections}) = P^*$

$p = \left(\frac{1}{500}\right)$ $p = (\frac{1}{4})$

$P(\text{not infected on first})$

$n = 100$ $n = 5$

$P(\text{not infected on second})$ ... not infected on 100th

$P(\text{not infected on 100th})$

$= 1 - (1 - \frac{1}{500})(1 - \frac{1}{500}) ... (1 - \frac{1}{500})$

$= 1 - \left(1 - \frac{1}{500}\right)^{100} = 0.18$

$P(\text{infected in 500 acts}) = 1 - \left(1 - \frac{1}{500}\right)^{500} = 0.63$

$P(\text{inf. in partner i in 1 act}) = p_i$

$P(\text{inf. in n acts}) = 1 - (1 - p_1)(1 - p_2)...(1 - p_n)$

(product distribution) $\prod_{i=1}^{n} (1-p_i)$ $\frac{1}{100} = p_i \leq \frac{1}{100}$

pretend all $p_i = p$ right answer: $1 - (1 - p)^n$

Dr.S. $nP$

Wolfram alpha: $nP = \frac{1}{2} ((n-1)n)p^2$ ... Taylor expansion

value is always smaller than $nP$
Lecture 3 (cont.)

Extra notes. (09/12/19)

- Order of infinity: 
  - Rational = integers
  - Countably infinite if 1-to-1 to integers
  - Irrational is one order of infinity than rational

- $\mathbb{N} = \{1, 2, 3, \ldots\}$ countable
- $\mathbb{R}$: all real numbers is uncountable

Measure theory: trying to make basic stuff rigorous
- Like length, area, volume
- But infinity is weird to become rigorous

Axiom of choice: If I have an uncountably infinite number of sets, I can make a new set by taking one element from each set.

Given a sphere, you can break up the sphere into a finite number of non-overlapping subsets, rotate them around, and assemble them into two identical copies of the original sphere.

When $S$ is infinite, the power set $2^S$ is "too big and strange" to permit the assignment of probabilities to all the $2^S$ sets in a logical, consistent way.

smaller set $C$ to restrict to subsets of $S$

1. $C$ includes entire sample space $S$
2. If event $A$ is in $C$, so is its complement $A^c$
3. If $A_1, A_2, \ldots, A_n$ are all in $C$, then so is: $\bigcup_{i=1}^{\infty} A_i$ union; add up, no duplicates.
Lecture 3 (cont.)
Whenever $|S| < \infty$ (finite), we can take $C = 2^S$, so we can meaningfully assign probabilities:

$(A^c)^c = A$

$\emptyset^c = S$ and $S^c = \emptyset$

$A \cup A = A$  \quad  $A \cup B = B \cup A$

$A \cup A^c = S$  \quad  \text{if } A \subseteq B, \text{ then } A \cup B = B$

$A \cup \emptyset = A$  \quad  $A \cup S = S$  \quad  \text{union} = "or"

**Associative property:** $A \cup (B \cup C) = (A \cup B) \cup C$

Intersection = "and"  \quad  $A^c \Rightarrow \text{not } A$  \quad  only elements

$A \cap B \Rightarrow A$ or $B$ \quad  belonging to

$A \cap B \Rightarrow A$ and $B$ \quad  both sets

**T-S case**

$A = \{\text{NNNNNNN}\}$ as a set is equivalent to the

true/false proposition: (exactly 0 T-S babies) being true

The intersection of $\bigcap_{i=1}^{n} A_i$ has the same

multiple sets:  \quad  associative property

Two sets $A, B$ are mutually exclusive $\Leftrightarrow$ disjoint

If $A \cap B = \emptyset$ (if they have no outcomes in common)

$n$ sets are disjoint if all distinct pairs

are disjoint: $A_i \cap A_j = \emptyset$ for $i \neq j$

mutually exclusive $\Rightarrow$ cannot both be true

De Morgan's Law  \quad  $(A \cup B)^c = A^c \cap B^c$

$(A \cap B)^c = A^c \cup B^c$

**Distributive property**  \quad  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$A \cup (B \cup C) = (A \cup B) \cup (A \cup C)$
AMS 131  Lecture 3 (cont.)

-  wake up sample space
-  all mutually exclusive
-  exhaustive, a point must be in one of the sections

\{ B_1, B_2, \ldots, B_n \} = \text{partition of } S.

Every outcome in } S \text{ lives in one (and only one) section of } S.

The partitions also apply to } A, \text{ where these sections are mutually exclusive and exhaustive}

\[ A \cup (A \cap B_1) \cup (A \cap B_2) \cup \ldots \cup (A \cap B_n) \]

\[ = (A \text{ and } B_1) \cup (A \text{ and } B_2) \cup \ldots \cup (A \text{ and } B_n) \]

\[ P(A) = P \left[ (A \text{ and } B_1) \cup (A \text{ and } B_2) \cup \ldots \cup (A \text{ and } B_n) \right] \]

\[ = P(A \text{ and } B_1) + P(A \text{ and } B_2) + \ldots + P(A \text{ and } B_n) \]

\[ = \sum_{i=1}^{n} P(A \text{ and } B_i) \quad \text{can sum them up, and there is no overlap!} \]

\[ P(A \text{ and } B_i) = P(B_i) \cdot P(A \mid B_i) \]

\[ = \sum_{i=1}^{n} P(B_i) \cdot P(A \mid B_i) \quad \text{Law of Total (LTP) \star} \]

- Problem 4 on take-home test #1

Kolmogorov Probability Axioms

\[ P_k(A) \]

\[ P_k(A) \text{ needs to be a function from } \mathcal{C} \text{ (the collection of non-weird subsets)} \text{ to the real number line } \mathbb{R} \text{ - what else should be assumed about } P_k? \]

\[ P_k(A) : \mathcal{C} \rightarrow [0,1] \quad \text{number (probability)} \]

"maps to" between 0 and 1
Lecture 3 (cont.)

Axiom 1: For all events $A, E, C$, $\Pr(A) \geq 0$ (motivated by relative frequency)

Axiom 2: $\Pr(S) = 1$

Axiom 3: For every countable collection of disjoint events $A_1, A_2, \ldots \in C$

$$\Pr(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \Pr(A_i)$$

add up all of the relative probabilities

Axiom 3': Equivalent to Axiom 3 - approximately if $A$ is "close" to $B$, then $\Pr(A) \approx \Pr(B)$

Consequences
1. $\Pr(\emptyset) = 0$
2. $\Pr(A^c) = 1 - \Pr(A)$
3. If $A \subset B$, then $\Pr(A) \leq \Pr(B)$
4. For all events $A$, $0 \leq \Pr(A) \leq 1$
5. For all events $A, B$ \hspace{1cm} $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$
   \hspace{1cm} general addition rule \hspace{1cm} "or" \hspace{1cm} "and"
6. For any events $A_1, A_2, \ldots, A_n$
   \hspace{1cm} $\Pr(\bigcap_{i=1}^{n} A_i) \geq \prod_{i=1}^{n} \Pr(A_i)$

$$\Pr(\bigcap_{i=1}^{n} A_i) \geq 1 - \sum_{i=1}^{n} \Pr(A_i^c)$$
   \hspace{1cm} Use $\cap$ in \hspace{1cm} starts.

T-S case \hspace{1cm} $\Pr(Y = 0) = \Pr(\text{exactly 0 T-S babies})$
\hspace{1cm} $= \Pr(1^{\text{st}} \text{ T-S and } 2^{\text{nd}} \not\text{ T-S} \ldots$
\hspace{1cm} $= [1 - \Pr(\text{baby})] \cdot [1 - \Pr(6^{\text{th}} \text{ T-S baby})] = 1 - p^9 (1 - p)^5$

$\Pr(\text{TITTT}) = p^9 \cdot 1p^9 (1-p)^0$
$\Pr(Y = 1) = 5p^1 (1-p)^4$
$\Pr(Y = 2) = 10p^2 (1-p)^3$

5 options for having 1 T-S baby \hspace{1cm} PASCALS Δ
Lecture 3 (cont.)

<table>
<thead>
<tr>
<th># of T-S banks</th>
<th>$p(Y = y)$</th>
<th>with $p = \frac{1}{2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$p^0(1-p)^0$</td>
<td>0.2573</td>
</tr>
<tr>
<td>1</td>
<td>$5p^1(1-p)^1$</td>
<td>0.3955</td>
</tr>
<tr>
<td>2</td>
<td>$10p^2(1-p)^2$</td>
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</tr>
<tr>
<td>3</td>
<td>$10p^3(1-p)^3$</td>
<td>0.0839</td>
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<tr>
<td>4</td>
<td>$5p^4(1-p)^4$</td>
<td>0.0141</td>
</tr>
<tr>
<td>5</td>
<td>$p^5(1-p)^5$</td>
<td>0.0010</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1.0000</td>
</tr>
</tbody>
</table>

$$P(Y = y) = \sum_{k=0}^{y} p^k(1-p)^{y-k}$$

**Pascal's Triangle**

*binomial coefficients: 1 5 10 10 5 1

Permutations and combinations: $n = 52$ cards

*no replacement*

$$\binom{52}{51} \cdot \frac{50}{49} \cdot \frac{48}{47} \cdot \ldots \cdot \frac{6}{5} \cdot \frac{1}{4} = 311,875,200$$

$(52), (51)$ ways for the first two permutations

The number of permutations of $n$ distinct things taken $k$ at a time is $P_{n,k} = \frac{n(n-1)\ldots(n-k+1)}{(n-k)!}$

How many orderings of 52 cards are there?

$$\binom{52}{51} \cdot \frac{50}{49} \cdot \frac{48}{47} \cdot \ldots \cdot \frac{6}{5} \cdot \frac{1}{4} = 311,875,200$$

$$P_{n,k} = \frac{n!}{(n-k)!}$$

$0! = 1$
T-S case where \( k = 1 \), one T-S baby
order does not matter! \( \{ T_1, N_1, N_2, N_3, N_4 \} \)
\( n! = 5! = 120 \) ways to order them.

Now, \( \{ T, N, N, N, N \} \)

\[ 1! \text{ to arrange } T, \ 4! \text{ to arrange } N \]s
\[ \frac{5!}{(1!)(4!)} = 5 = \frac{n!}{k!(n-k)!} \]

Given a set with \( n \) distinct elements, each distinct subset of size \( k \) is a combination
\[ C_{n,k} = \binom{n}{k} = \frac{n!}{k!(n-k)!} \]

\( n \) choose \( k \)

binomial coefficient

T-S case \( P(Y=y) = \binom{n}{y} p^y (1-p)^{n-y} \) valid for all \( y, n \geq 1 \)

\( P(\text{at least 2 people share the same birth date}) \)
- same month, day - not year
even birth rate from Jan 1 \( \rightarrow \) Dec 31, no Feb 29

365 days a year

\( k \) = \# of people registered \( k = 95 \) (AMS 72)

\( n^k \) equally likely outcomes \( n = 365 \)

If no one has same birthday:

\( \text{person 1} \ n = 365 \)
\( 2 \ n = 364 \)
\( \ldots \)
\( k \ n = n-k+1 \)

\[ P(\text{A}) = 1 - P(\text{not A}) \]

\[ P(\text{A}) = 1 - \frac{n!}{(n-k)! k!} \cdot 1 - \frac{365!}{(305)!(305)(305)\ldots(273)} \]

\[ = 0.999997 \]